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ROTARY OSCILLATIONS OF A RIGID DISC INCLUSION EMBEDDED IN AN ISOTROPIC ELASTIC INFINITE SPACE

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Abstract—This paper examines the asymmetric problem related to the harmonic oscillations of a rigid circular disc inclusion embedded in bonded contact with an isotropic elastic medium of infinite extent. The analysis of the problem is reduced to the solution of a single Fredholm integral equation of the second kind which is solved in an appropriate manner. The dynamic rotary stiffnesses are developed for a range of mass ratios and frequencies of practical interest.

1. INTRODUCTION

The class of problems which deals with the dynamic response of a rigid circular foundation resting on the surface of an elastic halfspace has received considerable attention. Results of such investigations have been particularly instrumental in improving the dynamic modelling of soil-structure interaction problems. Detailed accounts of these developments together with references to further work are given in the articles by Bycroft[1], Awojobi and Grootenhuis[2], Gladwell[3], Thomas[4], Luco and Westmann[5] and Richart *et al.*[6]. Further investigations relating to the dynamic behaviour of a rigid circular foundation resting on an elastic layer, a stratified medium and a non-homogeneous elastic medium are reported by Gladwell[7], Keer *et al.*[8] and Awojobi[9], respectively.

As is evident, a majority of these investigations concentrate on the dynamic behaviour of a disc resting on the surface of an elastic halfspace. The category of problems in which the rigid disc or inhomogeneity is embedded within the elastic medium has received only cursory attention. For example, problems related to rectilinear and torsional oscillations of a rigid sphere embedded in an elastic infinite space was examined by Chadwick and Trowbridge[10, 11]. Kanwal[12] and Williams[13] have examined the application of matched asymptotic expansion techniques to the analysis of rectilinear oscillations of an embedded

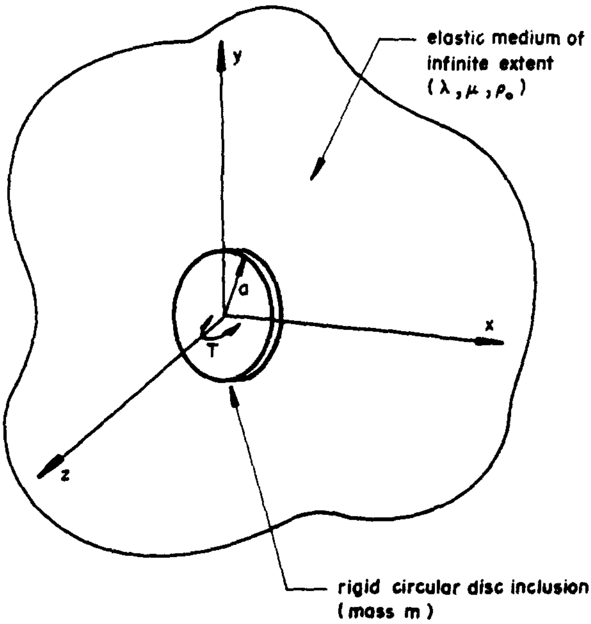

Fig. 1. Geometry of the embedded inclusion.

inclusion. More recently, Datta and Kanwal[14] have employed singularity methods to examine the problem concerning the rectilinear oscillations of a rigid spheroidal inclusion embedded in an elastic medium. In this particular paper we examine the asymmetric rotary oscillations of a disc shaped rigid inclusion embedded in bonded contact with an isotropic elastic medium of infinite extent (Fig. 1). The asymmetry of the deformation enables the reduction of the problem to a mixed boundary value problem associated with a halfspace region which is solved by means of integral transform techniques. Using a standard procedure, the problem is reduced to a Fredholm integral equation of the second kind which is solved in an approximate manner. Numerical results for the dynamic rotary stiffness of the embedded rigid disc inclusion are presented for a range of frequencies of practical interest.

2. SOLUTION

We consider the asymmetric deformation of the infinite space induced by the rotational oscillation of the bonded disc inclusion. The problem is antisymmetric in the normal stress σ_{zz}, the radial displacement u_r and the azimuthal displacement u_θ, about the plane $z = 0$, in the region $r \geq a$ for all $t \geq 0$. Also, since the rigid disc inclusion is bonded to the elastic medium, u_r and u_θ are zero in the region $z = 0$, $r \leq a$. Thus, we may restrict our attention to the analysis of a single halfspace region ($z \geq 0$) of the infinite space in which the plane $z = 0$ is subjected to the mixed boundary conditions

$$u_r(r, \theta, 0^+, t) = u_\theta(r, \theta, 0^+, t) = 0; \quad r \geq 0 \tag{1}$$

$$u_z(r, \theta, 0^+, t) = \phi r\, e^{i\omega t} \cos\theta; \qquad 0 \leq r \leq a \tag{2}$$

$$\sigma_{zz}(r, \theta, 0^+, t) = 0; \qquad a < r < \infty. \tag{3}$$

In addition, the displacement field should satisfy the radiation conditions appropriate for an infinite space region (see, e.g. Kupradze[15] and Eringen and Suhubi[16]). By making use of integral representations for the linear elastic equations of motion given by Bycroft[1], the boundary conditions (1)–(3) can be reduced to a system of dual integral equations. This dual system can be further reduced (see, e.g. Noble[17]) to a single Fredholm integral equation of the second kind

$$\Psi(t) + \frac{1}{\pi}\int_0^1 R(t, \tau)\Psi(\tau)\, d\tau = 2t \tag{4}$$

for the unknown function $\Psi(t)$. The function $R(t, \tau)$ is given by

$$R(t, \tau) = 2\int_0^\infty \left(\frac{4(1-\nu)}{(3-4\nu)k^2}\left\{\frac{\xi^3}{\beta} - \alpha\xi\right\} - 1\right)\sin(\xi t)\sin(\xi\tau)\, d\xi \tag{5}$$

where

$$\alpha = \begin{cases} [\xi^2 - h^2]^{1/2}; & \xi > h \\ -i[h^2 - \xi^2]^{1/2}; & 0 \leq \xi \leq h \end{cases} \tag{6}$$

$$\beta = \begin{cases} [\xi^2 - k^2]^{1/2}; & \xi > k \\ -i[k^2 - \xi^2]^{1/2}; & 0 \leq \xi \leq k \end{cases} \tag{7}$$

$$h^2 = \frac{\rho_0\omega^2a^2}{(\lambda + 2\mu)}; \quad k^2 = \frac{\rho_0\omega^2a^2}{\mu}. \tag{8}$$

ρ_0 is the equilibrium density and λ and μ are isentropic Lame's constants. The analysis of the embedded inclusion problem is reduced to the solution of the integral eqn (4). In the present paper we are primarily interested in evaluating the dynamic rotational stiffness of the embedded disc inclusion. It can be shown that the total torque acting on the embedded inclusion is given by

$$M(t)=\frac{32\phi\mu a^3(1-\nu)\,e^{i\omega t}}{(3-4\nu)}\int_0^1 t\Psi(t)\,dt. \tag{9}$$

Alternatively, by denoting the integral in (9) by $\int_0^1 t\Psi(t)\,dt=\psi_1+i\psi_2$ it can be shown that

$$M(t)=\frac{32\phi\mu a^3(1-\nu)}{(3-4\nu)}[\psi_1{}^2+\psi_2{}^2]^{1/2}\,e^{i(\omega t+\chi)} \tag{10}$$

where $\tan\chi=\psi_1/\psi_2$. In the statical case we have $\omega=0$; as a consequence $\psi_1=2t$ and $\psi_2=0$ and the expression (10) reduces to

$$M_0=\frac{64\mu a^3\phi(1-\nu)}{3(3-4\nu)}. \tag{11}$$

This result is in agreement with the expressions obtained by Kanwal and Sharma[18] and Selvadurai[19] for the static moment-rotation response for the embedded disc inclusion by making use of singularity methods and integral equation techniques respectively. The formal analysis can now be extended to include the effects of self weight of the embedded inclusion. In this particular instance, a rigid circular disc inclusion of mass m embedded in bonded contact with an elastic medium of infinite extent is subjected to a periodic rotation $\phi e^{i\omega t}$. The relationship between ϕ and the maximum amplitude T^* of the periodic couple $T(t)$ required to maintain the steady oscillation is given by

$$\frac{\phi a^3\mu}{T^*}=\frac{(3-4\nu)}{32(1-\nu)\left(\left\{\psi_1-\dfrac{k^2\Delta(3-4\nu)}{32(1-\nu)}\right\}^2+\psi_2{}^2\right)^{1/2}} \tag{12}$$

where $\Delta=m(1-\nu)/4\rho_0a^3$ is the mass ratio.

3. APPROXIMATE SOLUTION OF THE INTEGRAL EQUATION

For the solution of the integral eqn (4) we adopt a technique similar to that used by Robertson[20], Gladwell[3] and others for the analysis of equivalent forced vibration problems associated with a halfspace region. Avoiding details of calculation it can be shown that ψ_1 and ψ_2 can be expressed in the forms

$$\psi_1=\frac{2}{3}+\frac{k^2 4I_2}{15\pi}+k^4\left\{-\frac{4I_4}{35\pi}-\frac{34I_2{}^2}{315\pi^2}+k^6\right.$$

$$\left.\left\{\frac{8I_6}{189\pi}-\frac{2I_3{}^2}{27\pi^2}+\frac{47I_2{}^3}{2268\pi^3}+\frac{59I_2I_4}{2268\pi^2}\right\}+0(k^8)\right. \tag{13}$$

$$\psi_2=k^3\frac{2I_3}{9\pi}+k^5\left\{\frac{8I_2I_3}{45\pi^2}-\frac{2I_5}{45\pi}\right\}+0(k^7) \tag{14}$$

where

$$I_n=\frac{8(1-\nu)}{(3-4\nu)}\left(\int_0^\gamma\frac{x^n\{x^2+(\gamma^2-x^2)^{1/2}(1-x^2)^{1/2}\,dx}{(1-x^2)^{1/2}}+\int_\gamma^1\frac{x^{n+2}\,dx}{(1-x^2)^{1/2}}\right) \tag{15}$$

and

$$\gamma^2=\frac{(1-2\nu)}{(2-2\nu)}.$$

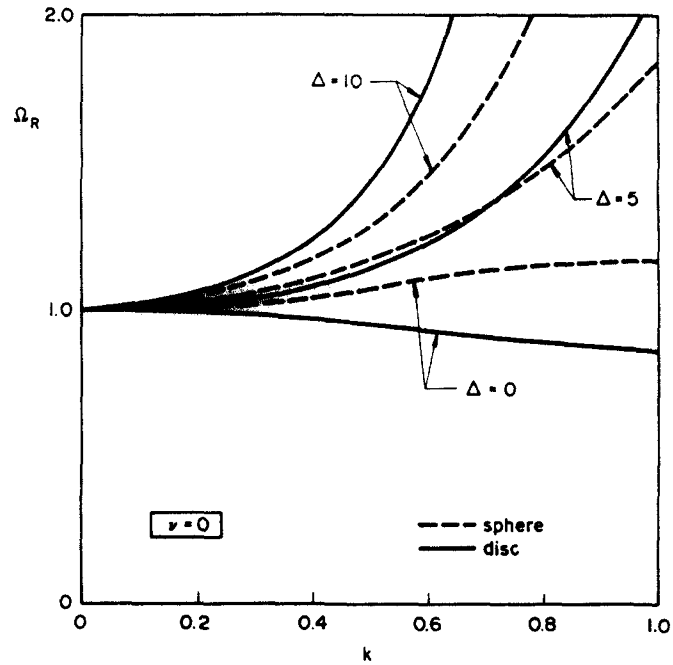

Fig. 2. Variation in the normalized dynamic rotational stiffness with non-dimensional frequency.

4. NUMERICAL RESULTS

In this section we present some numerical results which illustrate the manner in which the dynamic rotational compliance of the embedded rigid disc inclusion is influenced by the mass ratio (Δ) and the non-dimensional frequency k .The approximate analytical relationship used for this purpose is given by eqn (12) where ψ_1 and ψ_2 are defined by (13) and (14) respectively. Figures 2–4 illustrate typical variations in the normalized dynamic rotational stiffness $(\Omega_R)_{\text{disc}}$ which is obtained by making use of the result (12) for the dynamic rotational stiffness and the static equivalent defined by (11); i.e.

$$(\Omega_R)_{\text{disc}} = \frac{2}{3\left[\left\{\psi_1 - \dfrac{k^2\Delta(3-4\nu)}{32(1-\nu)^2}\right\}^2 + \psi_2^{\,2}\right]^{1/2}}. \tag{16}$$

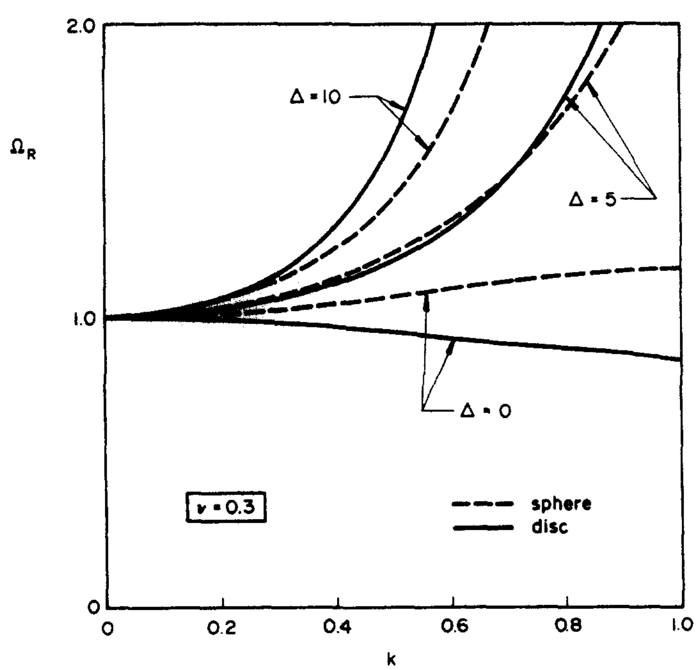

Fig. 3. Variation in the normalized dynamic rotational stiffness with non-dimensional frequency.

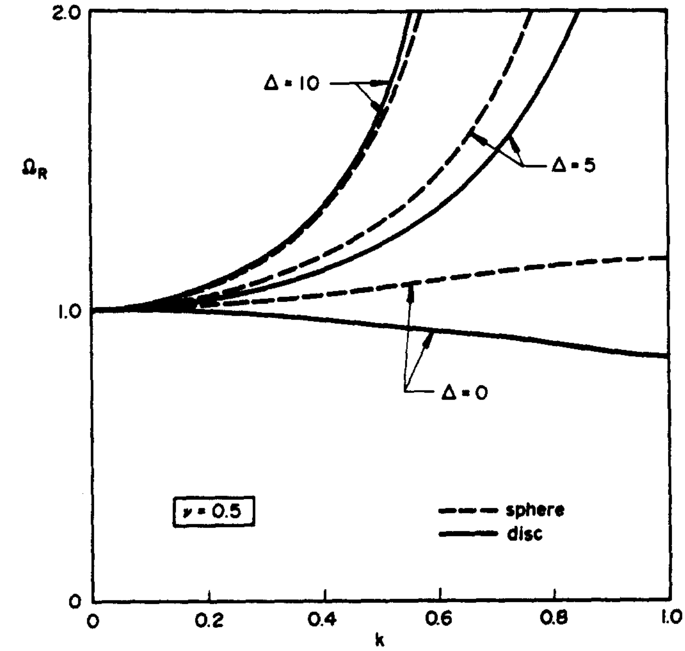

Fig. 4. Variation in the normalized dynamic rotational stiffness with non-dimensional frequency.

It is of interest to examine the correlation between the above approximation for the steady oscillation of the embedded disc inclusion and the equivalent result for the steady oscillation of a rigid sphere embedded in bonded contact with an elastic medium of infinite extent. The exact solution of the latter problem is given by Chadwick and Trowbridge[10]. A rigid sphere of mass m and radius a is subjected to a periodic rotation $\phi_s\, e^{i\omega t}$. The relationship between ϕ_s and the maximum amplitude (T_s^*) of the periodic force required to maintain the steady oscillation can be represented in the normalized form

$$(\Omega_R)_{\text{sphere}} = \frac{3\{1+k^2\}}{\left[\left\{3+2k^2-\dfrac{3\Delta(k^2+k^4)}{5\pi(1-\nu)}\right\}^2+k^6\right]^{1/2}} \tag{17}$$

and the static moment (T_{so}^*) rotation relationship is given by

$$T_{so}^* = 8\pi\phi_s\mu a^3. \tag{18}$$

REFERENCES

1. G. N. Bycroft, Forced vibrations of a rigid circular plate on a semi-infinite elastic space and on an elastic stratum. *Phil. Trans. Roy. Soc. Ser. A.* **248**, 327 (1956).
2. A. O. Awojobi and D. Grootenhuis, Vibration of rigid bodies on semi-infinite elastic media. *Proc. Roy. Soc. Ser. A.* **287**, 27 (1965).
3. G. M. L. Gladwell, Forced tangential and rotary vibration of a rigid circular disc on a semi-infinite solid. *Int. J. Engng Sci.* **6**, 591 (1968).
4. D. P. Thomas, Torsional oscillations of an elastic halfspace. *Q. J. Mech. Appl. Math.* **21**, 51 (1968).
5. J. E. Luco and R. A. Westmann, Dynamic response of circular footings. *J. Engng Mech. Div. Proc. A.S.C.E.* **97**, 1381 (1971).
6. F. E. Richart, J. R. Hall and R. D. Woods, *Vibrations of Soils and Foundations*. Prentice-Hall, New Jersey (1970).
7. G. M. L. Gladwell, Forced torsional vibrations of an elastic stratum. *Int. J. Engng Sci.* **7**, 1011 (1969).
8. L. M. Keer, H. H. Jabali and K. Chantaramungkorn, Torsional oscillations of a layer bonded to an elastic halfspace. *Int. J. Solids Structures* **10**, 1 (1974).
9. A. O. Awojobi, Vibration of rigid bodies on non-homogeneous semi-infinite elastic media. *Q. J. Mech. Appl. Math.* **26**, 483 (1973).
10. P. Chadwick and E. A. Trowbridge, Oscillations of a rigid sphere embedded in an infinite elastic solid. I. Torsional oscillations. *Proc. Camb. Phil. Soc.* **63**, 1189 (1967).
11. P. Chadwick and E. A. Trowbridge, Oscillations of a rigid sphere embedded in an infinite elastic solid. II. Rectilinear oscillations. *Proc. Camb. Phil. Soc.* **63**, 1207 (1967).
12. R. P. Kanwal, Dynamical displacements of an infinite elastic space and matched asymptotic expansions. *Quart. Appl. Math.* **44**, 275 (1965).

13. W. E. Williams, A note on the slow vibrations of an elastic medium. *Q. J. Mech. Appl. Math.* **19**, 1253 (1966).
14. S. Datta and R. P. Kanwal, Rectilinear oscillations of a rigid spheroid in an elastic medium. *Quart. Appl. Math.* **37**, 86 (1979).
15. V. D. Kupradze, Dynamical problems in elasticity. In *Progress in Solid Mechanics* (Edited by I. N. Sneddon and R. Hill), North Holland, Amsterdam (1963).
16. A. C. Eringen and E. Suhubi, *Elastodynamics: Linear Theory*, Vol. II. Academic Press, New York (1975).
17. B. Noble, The solution of Bessel function dual integral equations by a multiplying-factor method. *Proc. Camb. Phil. Soc.* **39**, 351 (1963).
18. R. P. Kanwal and D. L. Sharma, Singularity methods for elastostatics. *J. Elasticity* **6**, 405 (1976).
19. A. P. S. Selvadurai, The eccentric loading of a rigid circular foundation embedded in an isotropic elastic medium. *Int. J. Num. Anal. Meth. Goemech.* **4**, 121 (1980).
20. I. A. Robertson, Forced vertical vibration of a rigid circular disc on a semi-infinite elastic solid. *Proc. Camb. Phil. Soc.* **62**, 547 (1966).